Unlevered Beta

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The beta coefficient is a key parameter in the capital asset pricing model (CAPM). The beta coefficient measures the portion of volatility that cannot be diversified away. The risk-averse investor seeks to diversify away unsystematic risk, which is risk applicable to the individual stock. Systematic risk, which is risk applicable to the market as a whole, cannot be diversified away. Because unsystematic risk can be diversified away in the public markets investors are not compensated if they choose to employ suboptimal diversification. The equation for beta (β_s) as a function of the covariance of the individual stock's return (s) and the market portfolio return (m) and the variance of the market portfolio return (σ^2_m) is...

$$\beta_s = \frac{cov(s,m)}{\sigma^2_m} \tag{1}$$

The beta coefficient is used to calculate the risk-adjusted discount rate. This discount rate pays the investor the risk-free rate plus a premium for taking on systematic risk. The equation for discount rate (κ) as a function of the risk-free rate (r_f), the return on the market portfolio (r_m) and the beta coefficient (β_s) is...

$$\kappa = r_f + \beta_s \times (r_m - r_f) \tag{2}$$

The beta coefficients most often cited are equity betas. The equity beta is a function of both leverage and asset risk. Often times we want to calculate the asset beta, which is the equity beta before leverage. The equity beta is often referred to as the levered beta. The asset beta is often referred to as the unlevered beta. This paper will develop an equation that given the equity beta an asset beta can be derived.

Legend of Symbols

A_t	=	Market value of assets at time t (in dollars)
D_t	=	Market value of debt at time t (in dollars)
E_t	=	Market value of equity at time t (in dollars)
N_t	=	Number of shares outstanding at time t (in dollars)
S_t	=	Stock price at time t (in dollars)
W_t	=	Value of the driving Brownian motion at time t
μ_a	=	Expected annual rate of return on assets
μ_d	=	Expected annual rate of return on debt
μ_s	=	Expected annual rate of return on the stock
σ_a	=	Annual asset return volatility
σ_s	=	Annual stock return volatility

Equations for Asset Value, Debt Value and Stock Price

We will define the market value of assets (A_t) to be comprised of the market value of stockholders' equity, which is the number of shares outstanding (N_t) times the share price (S_t), plus the market value of debt (D_t). The equation for the market value of assets at time t is therefore...

$$A_t = N_t\, S_t + D_t \tag{3}$$

Given Equation (3) above the equation for the change in the market value of assets over the time interval $[t,\, t + \delta t]$ where δt is very small and near zero is...

$$\delta A_t = N_t\, \delta S_t + \delta D_t \tag{4}$$

The change in the market value of assets is defined by the following stochastic differential equation (SDE)...

$$\delta A_t = \mu_a A_t \, \delta t + \sigma_a A_t \, \delta W_t \tag{5}$$

Per the SDE in Equation (5) the change in the market value of assets is a function of an expected return, which is μ_a and can be estimated via the CAPM, and an unexpected return, which is a function of volatility σ_a and the change in the driving Brownian motion. Note that the Brownian motion W_t is normally-distributed with mean zero and variance $\sigma_a^2 t$.

The change in stock price is defined by the following stochastic differential equation (SDE)...

$$\delta S_t = \mu_s S_t \, \delta t + \sigma_s S_t \, \delta W_t \tag{6}$$

Per the SDE in Equation (6) the change in stock price is a function of an expected return, which is μ_s and can be estimated via the CAPM, and an unexpected return, which is a function of volatility σ_s and the change in the driving Brownian motion. Note that the Brownian motion W_t the drives the unexpected change in stock price in equation (6) is the same Brownian motion that drives the unexpected change in the market value of assets in Equation (5).

The change in the market value of debt is defined by the following ordinary differential equation (ODE)...

$$\delta D_t = \mu_d D_t \, \delta t \tag{7}$$

Per the ODE in Equation (7) the return on debt is a function of an expected return μ_d, which in this case is the market rate of interest.

Our goal is to solve for the two unknown variables which are (1) the expected return on assets (μ_a) and (2) the asset return volatility (σ_a). We will accomplish this task by equating the moments of the right and left hand sides of Equation (4) above.

Equating The First Moments

The first moment of Equation (4) is the expected value of Equation (4), which is...

$$\begin{aligned}
\mathbb{E}\Bigg[\delta A_t\Bigg] &= \mathbb{E}\Bigg[N_t \, \delta S_t + \delta D_t\Bigg] \\
&= \mathbb{E}\Bigg[N_t \, \delta S_t\Bigg] + \mathbb{E}\Bigg[\delta D_t\Bigg] \\
&= N_t \, \mathbb{E}\Bigg[\delta S_t\Bigg] + \mathbb{E}\Bigg[\delta D_t\Bigg]
\end{aligned} \tag{8}$$

If we substitute Appendix Equations (25), (27) and (29) for δA_t, δS_t and δD_t, respectively, in the equation above then we can rewrite Equation (8) as...

$$\mu_a A_t \, \delta t = N_t \, \mu_s S_t \, \delta t + \mu_d D_t \, \delta t \tag{9}$$

Given that the market value of stockholders' equity (E_t) is...

$$E_t = N_t \, S_t \tag{10}$$

Using the definition of stockholders' equity in Equation (10) we can rewrite Equation (9) as...

$$\mu_A A_t \, \delta t = \mu_S E_t \, \delta t + \mu_D D_t \, \delta t \tag{11}$$

Using Equation (11) above we can now solve for μ_a, which is the expected return on assets and our first unknown variable. The equation for the expected return on assets is...

$$\begin{aligned}
\mu_A &= \frac{\mu_S E_t \, \delta t}{A_t \, \delta t} + \frac{\mu_D D_t \, \delta t}{A_t \, \delta t} \\
&= \mu_S \frac{E_t}{A_t} + \mu_D \frac{D_t}{A_t}
\end{aligned} \tag{12}$$

Equating the Second Moments

The second moment of Equation (4) is the expected value of the square of Equation (4), which is...

$$\begin{aligned}
\mathbb{E}\Big[\delta A_t^2\Big] &= \mathbb{E}\Bigg[\bigg(N_t\,\delta S_t + \delta D_t\bigg)^2\Bigg] \\
&= \mathbb{E}\Big[N_t^2 \delta S_t^2 + 2\,N_t\,\delta S_t\,\delta D_t + \delta D_t^2\Big] \\
&= \mathbb{E}\Big[N_t^2 \delta S_t^2\Big] + \mathbb{E}\Big[2\,N_t\,\delta S_t\,\delta D_t\Big] + \mathbb{E}\Big[\delta D_t^2\Big] \\
&= N_t^2\,\mathbb{E}\Big[\delta S_t^2\Big] + 2\,N_t\,\mathbb{E}\Big[\delta S_t\,\delta D_t\Big] + \mathbb{E}\Big[\delta D_t^2\Big] \qquad (13)
\end{aligned}$$

If we substitute Appendix Equations (26), (28), (30) and (31) for δA_t^2, δS_t^2, δD_t^2 and $\delta S_t \delta D_t$, respectively, in the equation above then we can rewrite Equation (13) as...

$$\sigma_a^2 A_t^2 \delta t = N_t^2 \sigma_s^2 S_t^2 \delta t \qquad (14)$$

If we square the definition of equity in Equation (10) above then the square of equity is...

$$E_t^2 = N_t^2 S_t^2 \qquad (15)$$

Using the definition of the square of stockholders' equity in Equation (15) we can rewrite Equation (14) as..

$$\sigma_a^2 A_t^2 \delta t = \sigma_s^2 E_t^2 \delta t \qquad (16)$$

Using Equation (16) above we can now solve for σ_a, which is the volatility of asset returns and our second unknown variable. The equation for the volatility of asset returns is..

$$\begin{aligned}
\sigma_a^2 &= \frac{\sigma_s^2 E_t^2 \delta t}{A_t^2 \delta t} \\
\sigma_a^2 &= \sigma_s^2 \frac{E_t^2}{A_t^2} \\
\sigma_a &= \sigma_s \frac{E_t}{A_t} \qquad (17)
\end{aligned}$$

Unlevered Asset Beta

If we define ρ_{sm} to be the correlation between the individual stock's return and the market portfolio return then we can rewrite Equation (1) as...

$$\beta_s = \rho_{sm} \frac{\sigma_s}{\sigma_m} \qquad (18)$$

If we multiple both sides of Equation (18) by the ratio of the market value of stockholders' equity to the market value of assets then Equation (18) becomes...

$$\frac{E_t}{A_t} \beta_s = \rho_{sm} \frac{\sigma_s}{\sigma_m} \frac{E_t}{A_t} \qquad (19)$$

Using the definition of asset volatility as defined by Equation (17) we can rewrite Equation (19) as...

$$\frac{E_t}{A_t} \beta_s = \rho_{sm} \frac{\sigma_a}{\sigma_m} \qquad (20)$$

The correlation between the random return on stockholders' equity and the random return on the market portfolio is ρ_{sm}. The correlation between the random return on assets and the random return on the market portfolio is ρ_{am}. Note that the random return on assets is the random return on stockholders' equity plus a constant, which does not affect correlation. We can therefore make this definition...

$$\rho_{sm} = \rho_{am} \qquad (21)$$

Note that intuitively this makes sense since the Brownian motion that drives stock price is the same Brownian motion that drives asset value.

Using the definition as provided by Equation (21) we can rewrite Equation (20) as...

$$\frac{E_t}{A_t}\beta_s = \rho_{am}\frac{\sigma_a}{\sigma_m} \tag{22}$$

Note that per Equation (18) the left hand side of Equation (22) is the asset beta. We can therefore make this definition...

$$\beta_a = \frac{E_t}{A_t}\beta_s \tag{23}$$

A Hypothetical Problem

We are tasked with estimating the unlevered asset beta of Company ABC given the following parameters:

Parameters to the Problem:

Equity beta	=	1.35
Number shares outstanding	=	100
Share price in dollars	=	10.00
Book value of debt in dollars	=	400

Preliminary Calculations:

D_t	=	Market value of debt	=	400 (assume book value = market value)
E_t	=	Market value of equity	=	100 shares x 10.00 per share = 1000
A_t	=	Debt plus equity	=	1400

Problem Solution:

$$\beta_a = E_t \div A_t \times \beta_s \text{ [Equation (23)]}$$
$$\beta_a = 1000 \div 1400 \times 1.35$$
$$\beta_a = 0.96$$

Appendix

We will be using the following definitions...

$$\delta t^2 = 0 \;\; ...and... \;\; \delta W_t \delta t = 0 \;\; ...and... \;\; \delta W_t^2 = \delta t \tag{24}$$

The expectation of the change in asset value (Equation (5)) is...

$$\begin{aligned}\mathbb{E}\Big[\delta A_t\Big] &= \mathbb{E}\Big[\mu_a A_t\,\delta t + \sigma_a A_t\,\delta W_t\Big]\\ &= \mathbb{E}\Big[\mu_a A_t\,\delta t\Big] + \mathbb{E}\Big[\sigma_a A_t\,\delta W_t\Big]\\ &= \mu_a A_t \mathbb{E}\Big[\delta t\Big] + \sigma_a A_t \mathbb{E}\Big[\delta W_t\Big]\\ &= \mu_a A_t\,\delta t\end{aligned} \tag{25}$$

The expectation of the square of the change in asset value (Equation (5)) is...

$$\begin{aligned}
\mathbb{E}\left[\delta A_t^2\right] &= \mathbb{E}\left[\left(\mu_a A_t\,\delta t + \sigma_a A_t\,\delta W_t\right)^2\right] \\
&= \mathbb{E}\left[\mu_a^2 A_t^2 \delta t^2 + 2\,\mu_a \sigma_a A_t^2\,\delta t\,\delta W_t + \sigma_a^2 A_t^2 \delta W_t^2\right] \\
&= \mathbb{E}\left[\mu_a^2 A_t^2 \delta t^2\right] + \mathbb{E}\left[2\,\mu_a \sigma_a A_t^2\,\delta t\,\delta W_t\right] + \mathbb{E}\left[\sigma_a^2 A_t^2 \delta W_t^2\right] \\
&= \mu_a^2 A_t^2 \mathbb{E}\left[\delta t^2\right] + 2\,\mu_a \sigma_a A_t^2 \mathbb{E}\left[\delta t\,\delta W_t\right] + \sigma_a^2 A_t^2 \mathbb{E}\left[\delta W_t^2\right] \\
&= \sigma_a^2 A_t^2 \delta t
\end{aligned} \tag{26}$$

The expectation of the change in stock price (Equation (6)) is...

$$\begin{aligned}
\mathbb{E}\left[\delta S_t\right] &= \mathbb{E}\left[\mu_s S_t\,\delta t + \sigma_s S_t\,\delta W_t\right] \\
&= \mathbb{E}\left[\mu_s S_t\,\delta t\right] + \mathbb{E}\left[\sigma_s S_t\,\delta W_t\right] \\
&= \mu_s S_t \mathbb{E}\left[\delta t\right] + \sigma_s S_t \mathbb{E}\left[\delta W_t\right] \\
&= \mu_s S_t\,\delta t
\end{aligned} \tag{27}$$

The expectation of the square of the change in stock price (Equation (6)) is...

$$\begin{aligned}
\mathbb{E}\left[\delta S_t^2\right] &= \mathbb{E}\left[\left(\mu_s S_t\,\delta t + \sigma_s S_t\,\delta W_t\right)^2\right] \\
&= \mathbb{E}\left[\mu_s^2 S_t^2 \delta t^2 + 2\,\mu_s \sigma_s S_t^2\,\delta t\,\delta W_t + \sigma_s^2 S_t^2 \delta W_t^2\right] \\
&= \mathbb{E}\left[\mu_s^2 S_t^2 \delta t^2\right] + \mathbb{E}\left[2\,\mu_s \sigma_s S_t^2\,\delta t\,\delta W_t\right] + \mathbb{E}\left[\sigma_s^2 S_t^2 \delta W_t^2\right] \\
&= \mu_s^2 S_t^2 \mathbb{E}\left[\delta t^2\right] + 2\,\mu_s \sigma_s S_t^2 \mathbb{E}\left[\delta t\,\delta W_t\right] + \sigma_s^2 S_t^2 \mathbb{E}\left[\delta W_t^2\right] \\
&= \sigma_s^2 S_t^2 \delta t
\end{aligned} \tag{28}$$

The expectation of the change in debt value (Equation (7)) is...

$$\begin{aligned}
\mathbb{E}\left[\delta D_t\right] &= \mathbb{E}\left[\mu_d D_t\,\delta t\right] \\
&= \mu_d D_t\,\mathbb{E}\left[\delta t\right] \\
&= \mu_d D_t\,\delta t
\end{aligned} \tag{29}$$

The expectation of the square of the change in debt value (Equation (7)) is...

$$\begin{aligned}
\mathbb{E}\left[\delta D_t^2\right] &= \mathbb{E}\left[\mu_d^2 D_t^2 \delta t^2\right] \\
&= \mu_d^2 D_t^2\,\mathbb{E}\left[\delta t^2\right] \\
&= 0
\end{aligned} \tag{30}$$

The expectation of the product of the change in stock price and the change in debt value (Equation (6) and (7)) is...

$$
\begin{aligned}
\mathbb{E}\left[\delta S_t \, \delta D_t\right] &= \mathbb{E}\left[\left\{\mu_s S_t \, \delta t + \sigma_s S_t \, \delta W_t\right\}\left\{\mu_d D_t \, \delta t\right\}\right] \\
&= \mathbb{E}\left[\mu_s \, \mu_d \, S_t \, D_t \, \delta t^2 + \sigma_s \, \mu_d \, S_t \, D_t \, \delta W_t \, \delta t\right] \\
&= \mathbb{E}\left[\mu_s \, \mu_d \, S_t \, D_t \, \delta t^2\right] + \mathbb{E}\left[\sigma_s \, \mu_d \, S_t \, D_t \, \delta W_t \, \delta t\right] \\
&= \mu_s \, \mu_d \, S_t \, D_t \, \mathbb{E}\left[\delta t^2\right] + \sigma_s \, \mu_{dD} \, S_t \, D_t \, \mathbb{E}\left[\delta W_t \, \delta t\right] \\
&= 0
\end{aligned} \tag{31}
$$